GAUSSIAN PROCESSES
EXERCISE SHEET 7: LINEAR STRUCTURE OF GAUSSIANS

Exercise 1.

Since C is positive definite so is C^{-1}, and we have $\langle x, x\rangle_C > 0$ for all $x \neq 0$. It is clear that $\langle x, y\rangle_C$ is also bilinear and thus forms an inner product.

Let $e_1, \dots, e_n$ be an orthonormal basis w.r.t. the inner product $\langle \cdot, \cdot\rangle_C$. We have

$$\mathbb{E}\langle \overline{X}, e_j\rangle_C \langle \overline{X}, e_k\rangle_C = \mathbb{E}\langle \overline{X}, C^{-1}e_j\rangle \langle \overline{X}, C^{-1}e_k\rangle = \langle C^{-1}e_j, CC^{-1}e_k\rangle = \langle e_j, e_k\rangle_C = \delta_{j,k},$$

so $\overline{X}$ is a standard Gaussian w.r.t. the inner product $\langle \cdot, \cdot\rangle_C$. The linear symmetries of $\overline{X}$ are then the ones that fix the C-inner product. If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear and such that $\langle Ax, Ay\rangle_C = \langle x, y\rangle_C$, then in terms of the standard inner product this means that

$$\langle C^{-1/2}Ax, C^{-1/2}Ay\rangle = \langle Ax, C^{-1}Ay\rangle = \langle x, C^{-1}y\rangle = \langle C^{-1/2}x, C^{-1/2}y\rangle,$$

so we see that $C^{-1/2}A$ is orthogonal. In other words, $A = C^{1/2}UC^{-1/2}$ for some orthogonal matrix U. $\square$

Exercise 2.

Let us compute $\langle \phi_j, \phi_k\rangle = \sum_{\ell=1}^n \sin(\pi\frac{\ell j}{n+1})\sin(\pi\frac{\ell k}{n+1})$. By using $\sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}$ we see that the above equals

$$\frac{1}{2}\sum_{\ell=1}^n \big(\cos(\pi\frac{\ell(j-k)}{n+1}) - \cos(\pi\frac{\ell(j+k)}{n+1})\big) = \frac{1}{2}\Re\Big(\sum_{\ell=1}^n \big(e^{i\pi\frac{\ell(j-k)}{n+1}} - e^{i\pi\frac{\ell(j+k)}{n+1}}\big)\Big).$$

Let us compute $\Re\Big(\sum_{\ell=1}^n e^{i\pi\frac{\ell a}{n+1}}\Big)$. If $a = 0$, this is just n. Otherwise it equals

$$\Re\Big(\frac{e^{i\pi a}-1}{e^{i\pi\frac{a}{n+1}}-1} - 1\Big) = \Re\Big(\frac{((-1)^a-1)(e^{-i\pi\frac{a}{n+1}}-1)}{2-2\cos(\pi\frac{a}{n+1})} - 1\Big) = -\frac{1+(-1)^a}{2}.$$

Hence if $j = k$, then

$$\langle \phi_j, \phi_k\rangle = \frac{n + \frac{1+(-1)^{2j}}{2}}{2} = \frac{n+1}{2}$$

and if $j \neq k$, then

$$\langle \phi_j, \phi_k\rangle = \frac{-\frac{1+(-1)^{j-k}}{2} + \frac{1+(-1)^{j+k}}{2}}{2} = 0.$$

This means that if we let $\psi_j = \sqrt{\frac{2}{n+1}}\phi_j$, then ψ_j form an orthonormal basis in $\mathbb{R}^n$. Thus by rotational invariance we may write

$$W(k) = \sum_{j=1}^n A_j\psi_j(k) = \sum_{j=1}^n \sqrt{\frac{2}{n+1}}A_j \sin(\pi\frac{kj}{n+1}),$$

where A_j are i.i.d. $N(0,1)$ random variables. $\square$

Exercise 3.

Let us start by noting that

$$n = \mathbb{E}\Gamma(n)^2 = n^2 c_n^2$$

since $\phi_i(n) = 0$ for $i = 1, \dots, n-1$. Hence we must have $c_n = 1/\sqrt{n}$ and

$$\Gamma(j) = \frac{j}{\sqrt{n}} Z_n + \sum_{i=1}^{n-1} c_i Z_i \phi_i(j).$$

Let $C := (\min(i,j))_{i,j=1}^{n-1}$ be the covariance matrix of $(\Gamma(1), \Gamma(2), \dots, \Gamma(n))$. Then

$$\sum_{j,k=1}^{n} \min(j,k)\phi_i(j)\phi_i(k) = \langle \phi_i, C\phi_i \rangle = \mathbb{E}\langle \Gamma, \phi_i \rangle^2 = \frac{1}{n}\langle \psi, \phi_i \rangle^2 + c_i^2 \langle \phi_i, \phi_i \rangle^2,$$

where $\psi(j) = j$. Thus we get

$$c_i^2 \|\phi_i\|^4 = \sum_{j,k=1}^{n} (\min(j,k) - \frac{jk}{n})\phi_i(j)\phi_i(k).$$

Let us next look at the matrix $D := (\min(j,k) - \frac{jk}{n})_{j,k=1}^{n-1}$. We will show that ϕ_i are eigenvectors of D. This is easier to check for the inverse matrix of D, which turns out to be the following tridiagonal matrix:

$$D^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

To see this, note that

$$(DD^{-1})_{ij} = \sum_{k=1}^{n-1} D_{i,k} D^{-1}_{k,j} = \sum_{k=j-1}^{j+1} (\min(i,k) - \frac{ik}{n}) D^{-1}_{k,j}$$

with the definition of $D^{-1}_{k,j}$ extended to the cases $k = 0$ or $k = n$ by the same rule that $D^{-1}_{k,j}$ is -1 if $|k-j| = 1$, 2 if $k = j$ and 0 otherwise. Note that this does not change anything since $\min(i,k) - \frac{ik}{n} = 0$ if $k \in \{0, n\}$. Thus we get

$$(DD^{-1})_{ij} = -(\min(i,j-1) - \frac{i(j-1)}{n}) + 2(\min(i,j) - \frac{ij}{n}) - (\min(i,j+1) - \frac{i(j+1)}{n}).$$

If $i < j$, then we have

$$(DD^{-1})_{ij} = -(i - \frac{i(j-1)}{n}) + 2(i - \frac{ij}{n}) - (i - \frac{i(j+1)}{n}) = 0,$$

and the same for $i > j$ since the matrix is symmetric. On the other hand if $i = j$, then we get

$$(DD^{-1})_{ii} = -(i - 1 - \frac{i(i-1)}{n}) + 2(i - \frac{ii}{n}) - (i - \frac{i(i+1)}{n}) = 1,$$

and hence we see that DD^{-1} is indeed the identity.

Let us next compute $(D^{-1}\phi_i)(j)$. This is simply

$$\sum_{k=1}^{n-1} D^{-1}_{j,k}\phi_i(k) = -\phi_i(j-1) + 2\phi_i(j) - \phi_i(j+1),$$

which holds true also in the case $j \in \{1, n-1\}$ since $\phi_i(0) = \phi_i(n) = 0$. By standard trigonometric formulas we have

$$\begin{aligned}
-\phi_i(j-1) + 2\phi_i(j) - \phi_i(j+1) &= 2\sin(\frac{\pi ij}{n}) - \sin(\frac{\pi i(j-1)}{n}) - \sin(\frac{\pi i(j+1)}{n}) \\
&= 2\sin(\frac{\pi ij}{n}) - 2\sin(\frac{\pi ij}{n})\cos(\frac{\pi i}{n}) \\
&= 4\sin(\frac{\pi i}{2n})^2 \sin(\frac{\pi ij}{n}).
\end{aligned}$$

Thus ϕ_i is an eigenvector of D^{-1} with eigenvalue $4\sin(\frac{\pi i}{2n})^2$, which means that it is also an eigenvector of D with eigenvalue $4^{-1}\sin(\frac{\pi i}{2n})^{-2}$.

Returning to our main computation we thus have

$$c_i^2\|\varphi_i\|^4 = \langle \varphi_i, D\varphi_i\rangle = \frac{1}{4\sin(\frac{\pi i}{2n})^2}\|\varphi_i\|^2,$$

so that

$$c_i = \frac{1}{2\sin(\frac{\pi i}{2n})\|\varphi_i\|} = \frac{1}{\sqrt{2n}\sin(\frac{\pi i}{2n})},$$

since we know from the previous exercise that $\|\varphi_i\|^2 = \frac{n}{2}$. Hence we must have

$$\Gamma(j) = \frac{j}{\sqrt{n}} Z_n + \sum_{i=1}^{n-1} \frac{Z_i}{\sqrt{2n}\sin(\frac{\pi i}{2n})}\phi_i(j),$$

and it remains to check that Γ so defined actually has the right covariance $\mathbb{E}\Gamma(j)\Gamma(k) = \min(j,k)$. But we have

$$\begin{aligned}
\mathbb{E}\Gamma(j)\Gamma(k) &= \frac{jk}{n} + \sum_{i=1}^{n-1} \frac{1}{2n\sin(\frac{\pi i}{2n})^2}\phi_i(j)\phi_i(k) = \frac{jk}{n} + \frac{2}{n}\sum_{i=1}^{n-1}(D\phi_i)(j)\phi_i(k) \\
&= \frac{jk}{n} + \frac{2}{n}\sum_{i=1}^{n-1}\sum_{\ell=1}^{n-1} D_{j,\ell}\phi_i(\ell)\phi_i(k) = \frac{jk}{n} + \frac{2}{n}\sum_{\ell=1}^{n-1} D_{j,\ell}\sum_{i=1}^{n-1}\phi_\ell(i)\phi_k(i) \\
&= \frac{jk}{n} + \frac{2}{n}\sum_{\ell=1}^{n-1} D_{j,\ell}\|\phi_k\|^2\delta_{\ell,k} = \frac{jk}{n} + D_{j,k} = \min(j,k),
\end{aligned}$$

where we have used the fact that $\phi_i(j) = \phi_j(i)$ for all i, j and also the fact (shown in the previous exercise) that ϕ_j and ϕ_k are orthogonal if $j \neq k$.

Remark: In fact the matrix D is the covariance matrix of a Gaussian random walk conditioned to hit 0 at time n. The decomposition in the exercise thus shows that the Gaussian random walk can be expressed as a sum of this conditioned process and a linear process jZ_n multiplied by $\frac{1}{\sqrt{n}}$ in order to get the right variance at time n. □

Exercise 4.

Fix $h_1, \dots, h_n \in H$ and write, for each $h \in H$ and $N \geq 1$,

$$S_N(h) := \sum_{k=1}^{N} \xi_k \langle e_k, h \rangle.$$

We first note that $(S_N(h))_{N\geq 1}$ is Cauchy in $L^2(\Omega)$: for $M > N$,

$$\mathbb{E}\big[|S_M(h) - S_N(h)|^2\big] = \sum_{k=N+1}^{M} \langle e_k, h \rangle^2 \longrightarrow 0 \quad (N \to \infty),$$

since $\sum_{k=1}^{\infty} \langle e_k, h \rangle^2 = \|h\|^2 < \infty$. Thus the series $\sum_{k=1}^{\infty} \xi_k \langle e_k, h \rangle$ converges in $L^2(\Omega)$, and we set $\langle X, h \rangle := \lim_{N\to\infty} S_N(h)$.

Now compute covariances by passing to the L^2-limit of the partial sums. For any indices i, j and any $N, M \geq 1$,

$$\mathbb{E}\big[S_N(h_i) S_M(h_j)\big] = \sum_{k=1}^{\min(N,M)} \langle e_k, h_i \rangle \langle e_k, h_j \rangle,$$

because $\mathbb{E}[\xi_k \xi_\ell] = \delta_{k\ell}$. Letting $N, M \to \infty$ and using L^2-convergence yields

$$\mathbb{E}[\langle X, h_i \rangle] = \mathbb{E}[\langle X, h_j \rangle] = 0,$$

and that

$$\operatorname{Cov}\big(\langle X, h_i \rangle, \langle X, h_j \rangle\big) = \mathbb{E}[\langle X, h_i \rangle \langle X, h_j \rangle] = \sum_{k=1}^{\infty} \langle e_k, h_i \rangle \langle e_k, h_j \rangle = \langle h_i, h_j \rangle,$$

which depends only on the vectors h_i, h_j and not on the chosen orthonormal basis.

This completes the proof. □